

Chapter 11. Conic Sections

Question-1

Find the centre and radius of the following circles:

(i) $x^2 + y^2 = 1$

(ii) $x^2 + y^2 - 4x - 6y - 9 = 0$

(iii) $x^2 + y^2 - 8x - 6y - 24 = 0$

(iv) $3x^2 + 3y^2 + 4x - 4y - 4 = 0$

(v) $(x - 3)(x - 5) + (y - 7)(y - 1) = 0$

Solution:

(i) The general equation of circle is $x^2 + y^2 + 2gx + 2fy + c = 0$.

$$2g = 0, 2f = 0, c = -1$$

$$\therefore \text{centre is } (-g, -f) = (0, 0)$$

$$\text{Radius} = \sqrt{g^2 + f^2 - c} = \sqrt{0^2 + 0^2 + 1} = 1 \text{ units}$$

(ii) The general equation of circle is $x^2 + y^2 - 4x - 6y - 9 = 0$.

$$2g = -4, 2f = -6, c = -9$$

$$g = -2, f = -3$$

$$\therefore \text{centre is } (-g, -f) = (2, 3)$$

$$\text{Radius} = \sqrt{g^2 + f^2 - c} = \sqrt{(-2)^2 + (-3)^2 + 9} = \sqrt{22} \text{ units}$$

(iii) The general equation of circle is $x^2 + y^2 - 8x - 6y - 24 = 0$.

$$2g = -8, 2f = -6, c = -24$$

$$g = -4, f = -3$$

$$\therefore \text{centre is } (-g, -f) = (4, 3)$$

$$\text{Radius} = \sqrt{g^2 + f^2 - c} = \sqrt{(-4)^2 + (-3)^2 + 24} = \sqrt{49} = 7 = 1 \text{ units}$$

(iv) The general equation of circle is $3x^2 + 3y^2 + 4x - 4y - 4 = 0$.

$$2g = 4/3, 2f = -4/3, c = -4/3$$

$$\therefore \text{centre is } (-g, -f) = (-2/3, 2/3)$$

$$\text{Radius} = \sqrt{g^2 + f^2 - c} = \sqrt{\left(\frac{2}{3}\right)^2 + \left(-\frac{2}{3}\right)^2 + \frac{4}{3}} = \sqrt{\frac{8}{9} + \frac{4}{3}} = \sqrt{\frac{20}{9}} \text{ unit}$$

$$(v) (x - 3)(x - 5) + (y - 7)(y - 1) = 0$$

$$x^2 - 8x + 15 + y^2 - 8y + 7 = 0$$

The general equation of circle is $x^2 + y^2 - 8x - 8y + 22 = 0$.

$$2g = -8, 2f = -8, c = 22$$

$$\therefore \text{centre is } (-g, -f) = (4, 4)$$

$$\text{Radius} = \sqrt{g^2 + f^2 - c} = \sqrt{(-4)^2 + (-4)^2 - 22} = \sqrt{16 + 16 - 22} = \sqrt{10} \text{ units}$$



Question-2

For what values of a and b does the equation $(a - 2)x^2 + by^2 + (b - 2)xy + 4x + 4y - 1 = 0$ represent a circle? Write down the resulting equation of the circle.

Solution:

The general condition for a second degree equation to represent a circle is $x^2 + y^2 + 2gx + 2fy + c = 0$.

$$2fy + c = 0.$$

The equation $(a - 2)x^2 + by^2 + (b - 2)xy + 4x + 4y - 1 = 0$ when compared with the general equation we get,

Coefficient of x^2 = Coefficient of y^2

$$\text{Thus, } a - 2 = b \dots\dots\dots(i)$$

Coefficient of xy is zero (i.e.) $h = 0$

$$b - 2 = 0$$

$$b = 2 \dots\dots\dots(ii)$$

Substitute (ii) in (i)

$$a = b + 2 = 2 + 2 = 4$$

$\therefore$ The required equation of circle is $2x^2 + by^2 + 4x + 4y - 1 = 0$.

Question-3

Find the equation of the circle passing through the point (2,3) and having its centre at (1, 2).

Solution:

If the centre is (h, k) and radius is r , then the equation of the circle is

$$(x - h)^2 + (y - k)^2 = r^2$$

$$\text{Here } (h, k) = (1, 2)$$

$(x - 1)^2 + (y - 2)^2 = r^2$ is the required equation of the circle.

The circle passes through (2, 3).

$$\therefore (2 - 1)^2 + (3 - 2)^2 = r^2 \quad \therefore (1)^2 + (1)^2 = r^2 \quad \therefore r^2 = 2$$

$\therefore$ The required equation of the circle is $(x - 1)^2 + (y - 2)^2 = 2$.

Question-4

$x + 2y = 7$, $2x + y = 8$ are two diameters of a circle with radius 5 units. Find the equation of the circle.

Solution:

$$x + 2y = 7 \dots\dots\dots(i)$$

$$2x + y = 8 \dots\dots\dots(ii)$$

$$(i) \times 2 - (ii)$$

$$2x + 4y = 14$$

$$3y = 6$$

$$y = 2$$

$$\therefore x = 7 - 2y = 7 - 2(2) = 7 - 4 = 3$$

The two diameters of a circle intersect each other at a point which is the centre of the circle.

$\therefore (3, 2)$ is the centre of a circle.

$\therefore$ The equation of the circle $(x - 3)^2 + (y - 2)^2 = 5^2$.

Question-5

The area of a circle is 16π square units. If the centre of the circle is $(7, -3)$, find the equation of the circle.

Solution:

Area of a circle = 16π square units $\pi r^2 = 16\pi$ $r^2 = 16$

The equation of the circle is $(x - 7)^2 + (y + 3)^2 = 16$.

Question-6

Find the equation of the circle whose centre is $(-4, 5)$ and circumference is 8π units.

Solution:

Circumference of a circle = 8π units

$$2\pi r = 16\pi \quad 2r = 16$$

$$r = 8 \text{ units}$$

The equation of the circle is $(x + 4)^2 + (y - 5)^2 = 64$.

Question-7

Find the circumference and area of the circle $x^2 + y^2 - 6x - 8y + 15 = 0$.

Solution:

$$x^2 + y^2 - 6x - 8y + 15 = 0$$

$$2g = -6, 2f = -8, c = 15$$

$$g = -3, f = -4$$

$$r^2 = g^2 + f^2 - c = 9 + 16 - 15 = 10$$

Area of the circle = $\pi r^2 = 10\pi$ square units

Circumference of the circle = $2\pi r = 2\pi\sqrt{10}$ units

Question-8

Find the equation of the circle which passes through $(2, 3)$ and whose centre is on x-axis and radius is 5 units.

Solution:

Centre is on x-axis.

$\therefore$ The equation of the circle is $(x - h)^2 + y^2 = r^2$.

$$(2 - h)^2 + 3^2 = 5^2$$

$$4 - 4h + h^2 + 9 = 25$$

$$h^2 - 4h - 12 = 0$$

$$(h - 6)(h + 2) = 0$$

$$h = 6, -2$$

$$(x - 6)^2 + y^2 = 25 \quad \text{or} \quad (x + 2)^2 + y^2 = 25$$

$$x^2 - 12x + 36 + y^2 = 25 \quad \text{or} \quad x^2 + 4x + 4 + y^2 = 25$$

$x^2 - 12x + y^2 + 11 = 0$ or $x^2 + 4x + y^2 - 21 = 0$ are the equations of the circle.



Question-9

Find the equation of the circle described on the line joining the points (1, 2) and (2, 4) as its diameter.

Solution:

The equation of the circle is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$.

Here $(x_1, y_1) = (1, 2)$ and $(x_2, y_2) = (2, 4)$

$$\therefore (x - 1)(x - 2) + (y - 2)(y - 4) = 0.$$

$$x^2 + y^2 - 3x - 6y + 10 = 0.$$

Question-10

Find the equation of the circle passing through the points (1, 0), (0, -1) and (0,1).

Solution:

The general equation of circle is $x^2 + y^2 + 2gx + 2fy + c = 0$.

The points (1, 0), (0, -1) and (0,1) lie on the circle.

If (1, 0) lie on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, we get,

$$\therefore 1 + 2g + c = 0$$

$$2g + c = -1 \dots\dots\dots(i)$$

If (0, -1) lie on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, we get,

$$1 - 2f + c = 0$$

$$-2f + c = -1 \dots\dots\dots(ii)$$

If (0, 1) lie on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, we get,

$$1 + 2f + c = 0$$

$$-2f + c = -1 \dots\dots\dots(ii)$$

If (0, 1) lie on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, we get,

$$1 + 2f + c = 0$$

$$2f + c = -1 \dots\dots\dots(iii)$$

$$(ii) + (iii)$$

$$2c = -2$$

$$c = -1$$

Substituting $c = -1$ in (i)

$$2g - 1 = 1$$

$$g = 1$$

Substituting $c = -1$ in (ii)

$$-2f - 1 = -1$$

$$f = 0$$

The general equation of circle is $x^2 + y^2 + 2x - 1 = 0$.

Question-11

Find the equation of the circle passing through the points (1, 1), (2, -1) and (3, 2).

Solution:

The general equation of circle is $x^2 + y^2 + 2gx + 2fy + c = 0$.

The points (1, 1), (2, -1) and (3, 2) lie on the circle.

$$\therefore 1 + 1 + 2g + 2f + c = 0$$

$$2g + 2f + c = -2 \dots\dots\dots(i)$$

$$4 + 1 + 4g - 2f + c = 0$$

$$4g - 2f + c = -5 \dots\dots\dots(ii)$$

$$9 + 4 + 6g + 4f + c = 0$$

$$6g + 4f + c = -13 \dots\dots\dots(iii)$$

$$(i) - (ii)$$

$$-2g + 4f = 3 \dots\dots\dots(iv)$$

$$(i) - (iii)$$

$$-4g - 2f = 11 \dots\dots\dots(v)$$

$$(iv) - [2(v)]$$

$$-10g = 25$$

$$g = \frac{25}{-10} = \frac{-5}{2}$$

$$\text{Substitute } g = \frac{-5}{2} \text{ in (iv)}$$

$$-2g + 4f = 3$$

$$-2\left(\frac{-5}{2}\right) + 4f = 3$$

$$5 + 4f = 3$$

$$4f = 3 - 5$$

$$= -2$$

$$f = \frac{-2}{4} = \frac{-1}{2}$$

$$\text{Substitute } g = \frac{-5}{2} \text{ and } f = \frac{-1}{2} \text{ in (i)}$$

$$2g + 2f + c = -2$$

$$2\left(\frac{-5}{2}\right) + 2\left(\frac{-1}{2}\right) + c = -2$$

$$-5 + (-1) + c = -2$$

$$c = -2 + 6 = 4$$

By substituting $g = \frac{-5}{2}$, $f = \frac{-1}{2}$ and $c = 4$ in $x^2 + y^2 + 2gx + 2fy + c = 0$ we get,

$\therefore$ The general equation of circle is $x^2 + y^2 - 5x - y + 4 = 0$.

Question-12

Find the equation of the circle that passes through the points (4, 1) and (6, 5) and has its centre on the line $4x + y = 16$.

Solution:

The general equation of the circle is $x^2 + y^2 + 2gx + 2fy + c = 0$

(4, 1) lies on the circle.

$$16 + 1 + 8g + 2f + c = 0$$

$$8g + 2f + c = -17 \dots\dots\dots (i)$$

(6, 5) lies on the circle.

$$36 + 25 + 12g + 10f + c = 0$$

$$12g + 10f + c = -61 \dots\dots\dots (ii)$$

$$(ii) - (i)$$

$$4g + 8f = -44$$

$$g + 2f = -11 \dots\dots\dots (iii)$$

Let $(-g, -f)$ be the centre of the circle lying on $4x + y = 16$.

$$\therefore -4g - f = 16 \dots\dots\dots (iv)$$

$$2(iv) + (iii)$$

$$-7g = 21$$

$$g = -3$$

Substitute $g = -3$ in (iv)

$$-4(-3) - f = 16$$

$$f = -4$$

Substitute in (i)

$$8(-3) + 2(-4) + c = -17$$

$$-24 - 8 + c = -17$$

$$c = -17 + 32$$

$$c = 15 \therefore \text{The general equation of the circle is } x^2 + y^2 - 6x - 8y + 15 = 0.$$

Question-13

Find the equation of the circle whose centre is on the line $x = 2y$ and which passes through the points $(-1, 2)$ and $(3, -2)$.

Solution:

The general equation of the circle is $x^2 + y^2 + 2gx + 2fy + c = 0$. $(-g, -f)$ is centre of the circle.

Centre $(-g, -f)$ lies on the line $x = 2y$.

$$\therefore -g = -2f$$

$$g = 2f \dots\dots\dots(i)$$

$(-1, 2)$ lies on the circle.

$$\therefore 1 + 4 - 2g + 4f + c = 0$$

$$1 + 4 - 2(2f) + 4f + c = 0 \text{ (from i)}$$

$$\therefore c = -5 \dots\dots\dots(ii)$$

$(3, -2)$ lies on the circle.

$$\therefore 9 + 4 + 6g - 4f + c = 0$$

$$\therefore 9 + 4 + 6(2f) - 4f - 5 = 0 \text{ (from i and ii)}$$

$$8f + 8 = 0$$

$$f = -1 \dots\dots\dots(iii)$$

$$g = -2 \dots\dots\dots(iv) \therefore \text{The required equation of the circle is } x^2 + y^2 - 4x - 2y - 5 = 0.$$

Question-14

Find the cartesian equation of the circle whose parametric equations are $x = \frac{1}{4} \cos \theta$, $y = \frac{1}{4} \sin \theta$ and $0 \leq \theta \leq 2\pi$.

Solution:

$$\cos \theta = 4x$$

$$\sin \theta = 4y$$

$$\cos^2 \theta + \sin^2 \theta = 1$$

$$\therefore 16x^2 + 16y^2 = 1 \text{ is the required cartesian equation of the circle.}$$

Question-15

Find the parametric equation of the circle $4x^2 + 4y^2 = 9$.

Solution:

$$r^2 = 9/4$$

$$\therefore r = 3/2$$

$\therefore$ The parametric equations of the given circle $4x^2 + 4y^2 = 9$ are

$$x = \frac{3}{2} \cos \theta \text{ and } y = \frac{3}{2} \sin \theta, 0 \leq \theta \leq 2\pi.$$

Question-16

Find the coordinates of foci, equations of the directrices, and the length of the latusrectum of the parabola: $y^2 = 12x$.

Solution:

The given problem $y^2 = 12x$ is of the form $y^2 = 4ax$, where $4a = 12$. (i.e.) $a = 3$. The coordinates of the focus $(a, 0)$ is $(3, 0)$. The equation of the directrix is $x = -a$ (i.e.) $(-3, 0)$ Length of the latusrectum $= 4a = 12$.

Question-17

Find the foci, vertices of the parabola: $y = -4x^2 + 3x$.

Solution:

$$y = -4x^2 + 3x$$

$$4x^2 - 3x = -y$$

$$x^2 - \frac{3}{4}x = \frac{-y}{4}$$

$$x^2 - 2x\left(\frac{3}{8}\right) + \left(\frac{3}{8}\right)^2 = \frac{-y}{4} + \left(\frac{3}{8}\right)^2$$

$$\left(x - \frac{3}{8}\right)^2 = -1\left[\frac{y}{4} - \left(\frac{3}{8}\right)^2\right]$$

$$= \frac{-1}{4}\left[y - \frac{9}{16}\right]$$

$$\left(x - \frac{3}{8}\right)^2 = 4 \times \left(\frac{-1}{16}\right)\left(y - \frac{9}{16}\right) \dots\dots\dots (i)$$

Shifting the origin to the point $\left(\frac{3}{8}, \frac{9}{16}\right)$ without rotating the axes and denoting the new coordinates w.r.t. these axes by X and Y we have,

$$\text{Put } X = \left(x - \frac{3}{8}\right)$$

$$Y = \left(y - \frac{9}{16}\right)$$

In original coordinates :vertex is $X = 0$

$$\left(x - \frac{3}{8}\right) = 0 \Rightarrow x = \frac{3}{8}$$

$$Y = 0$$

$$\left(y - \frac{9}{16}\right) = 0 \Rightarrow y = \frac{9}{16}$$

$$\therefore \text{Vertex is } \left(\frac{3}{8}, \frac{9}{16}\right)$$

Focus: The coordinates of the focus with respect to the new axes are $(X = 0, Y = a)$

In original coordinates,

$$\text{Focus } \left(x - \frac{3}{8}\right) = 0 \Rightarrow x = \frac{3}{8}$$

$$\left(y - \frac{9}{16}\right) = \frac{-1}{16}$$

$$y = \left(\frac{9}{16} - \frac{1}{16}\right) = \frac{8}{16} = \frac{1}{2}$$

$$\therefore \text{Focus is } \left(\frac{3}{8}, \frac{1}{2}\right)$$

Question-18

Find the equation of the parabola whose directrix is $x = 0$ and focus at $(6, 0)$.

Solution:

Let $P(x, y)$ be a point on the parabola. Join SP .

$$SP = \sqrt{(x-6)^2 + (y-0)^2} = \sqrt{x^2 - 12x + 36 + y^2}$$

$$PM = x$$

$$SP = PM$$

$$x^2 = (x-6)^2 + y^2$$

$$x^2 = x^2 - 12x + 36 + y^2$$

$$y^2 - 12x + 36 = 0$$

$$y^2 = 12x - 36$$

Question-19

Does the point $(7, -11)$ lie inside or outside the circle $x^2 + y^2 - 10x = 0$?

Solution:

By substituting the point $(7, -11)$ in the equation $x^2 + y^2 - 10x$, we get

$$7^2 + (-11)^2 - 10(7) = 49 + 121 - 70$$

$$= 170 - 70 = 100 > 0$$

Thus the point $(7, -11)$ lies outside the circle.

Question-20

Determine whether the points $(-2, 1)$, $(0, 0)$ and $(4, -3)$ lie outside, on or inside the circle $x^2 + y^2 - 5x + 2y - 5 = 0$.

Solution:

By substituting $(-2, 1)$ in the equation $x^2 + y^2 - 5x + 2y - 5$ we get,

$$\begin{aligned} (-2)^2 + (1)^2 - 5(-2) + 2(1) - 5 &= 4 + 1 + 10 + 2 - 5 \\ &= 12 > 0. \end{aligned}$$

Thus the point $(0, 0)$ lie on the circle .

By substituting $(0, 0)$ in the equation $x^2 + y^2 - 5x + 2y - 5$ we get,

$$\begin{aligned} (0)^2 + (0)^2 - 5(0) + 2(0) - 5 &= 0 + 0 + 0 + 2 - 5 \\ &= -5 < 0. \end{aligned}$$

Thus the point $(0, 0)$ lie inside the circle.

By substituting $(4, -3)$ in the equation $x^2 + y^2 - 5x + 2y - 5$ we get,

$$\begin{aligned} (4)^2 + (-3)^2 - 5(4) + 2(-3) - 5 &= 16 + 9 + 20 - 6 - 5 \\ &= 34 > 0. \end{aligned}$$

Thus the point $(4, -3)$ lie on the circle.



Question-21

For the following ellipses find the lengths of major and minor axes, coordinates of foci and vertices and eccentricity $3x^2 + 2y^2 = 6$.

Solution:

$$3x^2 + 2y^2 = 6.$$

Dividing by 6 we get,

$$\frac{3x^2}{6} + \frac{2y^2}{6} = 1$$

$$\Rightarrow \frac{x^2}{2} + \frac{y^2}{3} = 1$$

This equation is of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $a^2 = 2$ and $b^2 = 3$. (i.e.,) $a = \sqrt{2}$ and $b = \sqrt{3}$

Here $a < b$, so the major and minor axes of the given ellipse are along y and x – axes respectively.

Length of the major axis = $2b = 2\sqrt{3}$, Length of the minor axis = $2a = 2\sqrt{2}$.

The coordinates of the vertices are $(0, b)$ and $(0, -b)$.

(i.e.,) $(0, \sqrt{3})$ and $(0, -\sqrt{3})$

The eccentricity e of the ellipse is given by $e = \sqrt{1 - \frac{a^2}{b^2}} = \sqrt{1 - \frac{2}{3}} = \frac{1}{\sqrt{3}}$.

The coordinates of the foci are $(0, be)$ and $(0, -be)$ (i.e.,) $(0, 1)$ and $(0, -1)$.

Question-22

The foci of an ellipse are $(\pm 8, 0)$ and its eccentricity is $\frac{1}{2}$, find its equation.

Solution:

Let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. The coordinates of foci are $(\pm ae, 0)$.

$$ae = 8$$

$$a \times \frac{1}{2} = 8$$

$$\Rightarrow a = 16$$

Thus

$$b^2 = a^2 (1 - e^2)$$

$$= 256 \left(1 - \frac{1}{4}\right)$$

$$= 192$$

Hence, the equation of the ellipse is $\frac{x^2}{256} + \frac{y^2}{192} = 1$



Question-23

Find the equation of the ellipse whose foci are (4, 3), (-4, 3) and whose semi –minor axis is 3.

Solution:

Let S and S' be two foci of the required ellipse. Then the coordinates of S and S' are (4, 3) and (-4, 3) respectively.

$$SS' = 8$$

Let 2a and 2b be the lengths of the axes of the ellipse and e be the eccentricity.

Then $SS' = 2ae$, but $2ae = 8$

Thus, $ae = 4$.

$$\text{Now, } b^2 = a^2(1 - e^2)$$

$$9 = a^2 - 4^2$$

$$9 + 16 = a^2$$

$$a = 5$$

Let P(x, y) be any point on the ellipse.

Then,

$$SP + S'P = 2a$$

$$\sqrt{(x-4)^2 + (y-3)^2} + \sqrt{(x+4)^2 + (y-3)^2} = 10$$

$$\left[(x-4)^2 + (y-3)^2 \right] - \left[(x+4)^2 + (y-3)^2 \right] = 100 - 20\sqrt{(x+4)^2 + (y-3)^2}$$

$$-16x = 100 - 20\sqrt{(x+4)^2 + (y-3)^2}$$

$$16x + 100 = 20\sqrt{(x+4)^2 + (y-3)^2}$$

Dividing by 4 we get,

$$4x + 25 = 5\sqrt{(x+4)^2 + (y-3)^2}$$

Squaring we get,

$$(4x + 25)^2 = 25 \left[(x+4)^2 + (y-3)^2 \right]$$

$$16x^2 + 200x + 625 = 25x^2 + 200x + 400 + 25y^2 - 150y + 225$$

$$9x^2 + 25y^2 - 150y = 0.$$

Question-24

Solution:

(i) Let P be the point on x-axis where it touches the circle.

Centre C is (5, 6) and P is (5, 0).

$$r = CP = \sqrt{(5-5)^2 + (6-0)^2} = 6.$$

The equation of the circle is $(x-5)^2 + (y-6)^2 = 6^2$.

$$x^2 + y^2 - 10x - 12y + 25 + 36 = 36$$

$$x^2 + y^2 - 10x - 12y + 25 = 0$$

(ii) Let P be the point on y-axis where it touches the circle.

Centre C is (5, 6) and P is (0, 6).

$$r = CP = \sqrt{(5-0)^2 + (6-6)^2} = 5.$$

The equation of the circle is $(x-5)^2 + (y-6)^2 = 5^2$.

$$x^2 + y^2 - 10x - 12y + 36 = 0$$

